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Stationary-state skewness in KPZ-type growth

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Abstract. Stationary states in KPZ-type growth have interesting short distance properties. We find that typically they are skewed and lack particle-hole symmetry. For example, hill-tops are typically flatter than valley-bottoms, and all odd moments of the height distribution function are non-zero. Stationary-state skewness can be turned on and off in the $(1+1)$-dimensional restricted solid-on-solid (RSOS) model. We construct the exact stationary state for its master equation in a four-dimensional parameter space. In this state steps are completely uncorrelated. Familiar models such as the Kim–Kosterlitz model lie outside this space, and their stationary states are skewed. We demonstrate using finite size scaling that the skewness diverges with systems size, but such that the skewness operator is irrelevant in $(1+1)$ dimensions, with an exponent $y_{sk} \simeq -1$, and that the KPZ fixed point lies at zero-skewness.

1. Introduction

Crystal surfaces display interesting scaling properties during growth. One of the dynamic processes that has been at the centre of attention is the so-called KPZ-type growth, named after the Langevin equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = v_0 + \nu\nabla^2 h + \tfrac{1}{2}\lambda(\nabla h)^2 + \eta \tag{1.1a}$$

with uncorrelated Gaussian noise

$$\langle \eta(\boldsymbol{r}_1, t_1)\eta(\boldsymbol{r}_2, t_2)\rangle = 2D\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2)\delta(t_1 - t_2) \tag{1.1b}$$

studied by Kardar *et al* [1]. The growth rate, v_0, is modified by the local curvature of the surface (the ν-term), its local slope (the λ-term), and random fluctuations (the η-term). It is well established by now that many microscopic growth processes belong to the KPZ universality class [2–5]. Such microscopic models have been investigated by numerous Monte Carlo (MC) simulations and finite size scaling (FSS) studies using exact diagonalization of master equations. Moreover, at least one model is exactly soluble in $(1+1)$ dimensions ($D = 1$) [6–9]. The exact stationary state for several solid-on-solid models has been found as well [10–13]. Intriguing relations with different aspects of physics have been established. Equation (1.1) is equivalent to the Burgers equation for randomly stirred fluids (∇h represents the fluid velocity) [14]. KPZ growth maps onto the directed polymer problem [15, 5], and relates to equilibrium liquid crystal phases [16]. In one dimension (1D) it describes persistent currents in metal rings in the context of asymmetric exclusion models [10, 11, 17–19], and is also equivalent to the equilibrium statistical mechanics at facet-ridge endpoints in two dimensional (2D) crystals [17].

The scaling properties of KPZ-type growth have been established, but our understanding is not yet at the level we would like. It is useful to make a comparison with equilibrium

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critical phenomena. Master equations for microscopic models, such as the body-centred solid-on-solid (BCSOS) model and the restricted solid-on-solid (RSOS) model, play a similar role as Ising-type microscopic models in equilibrium critical phenomena. KPZ-type Langevin equations play a similar role as ϕ^4-type field theories. There are roughly three levels at which a particular type of scaling invariance can be understood. The first level is to establish empirically (experimentally and numerically) the existence of scale invariance and universality. For equilibrium critical phenomena this was achieved in the early 1970s, for KPZ-type growth only a few years ago.

The second level is to obtain analytical confirmation of the empirical scaling properties. Exactly soluble models confirm the empirical scaling properties of 2D equilibrium critical phenomena. Mean-field approximations and Landau–Ginzburg theory yield the existence of an upper critical dimension and makes it possible to carry out controlled renormalization transformations that demonstrate the existence of the Ising fixed point in ϕ^4 theory. For KPZ-type growth, we have only reached this level in 1D. The BCSOS growth model is exactly soluble in 1D. The KPZ fixed point in general dimension (D) has eluded us thus far. The existence of an upper critical dimension is yet unclear. The ϵ-expansion analysis of equation (1.1) [1, 14, 20] describes the reversal of the stability of Edwards–Wilkinson (EW) growth (the point $\lambda = 0$) with respect to λ at $D = 2$. It does not yield a fixed point for the KPZ universality class, except a strong-coupling one in 1D.

The third level is to express the scaling properties in terms of a free-field theory. In general this is not possible at all. However, for 2D equilibrium critical phenomena such a description emerged during the last 15 yr. Coulomb gas representations [21, 22] and conformal field theory [23] provide a full free-field theory description of virtually all 2D equilibrium phase transitions. There is no assurance that this can be generalized to dynamic processes in 1D, but we have a good chance since the time-evolution operators of master equations in $(1+1)$ dimensions closely resemble transfer matrices of 2D equilibrium critical phenomena. Generalizations of conformal invariance are being considered [24, 25], but it is too early to tell whether this will work.

In this paper we present a master equation study of a generalized RSOS growth model. In this model, nearest-neighbour columns of particles can only differ by $dh = 0, \pm 1$. We consider the most general growth rule involving only nearest-neighbour step configurations. This gives rise to a five-dimensional (5D) phase diagram. We address four issues: (a) we describe the global structure of this phase diagram; (b) we derive the exact analytic form of the stationary state in a four-dimensional (4D) subspace; (c) we point out that the stationary state is generically skewed; and (d) we investigate whether stationary state skewness scales in accordance with conventional renormalization theory and the concept of universality.

Somehow, stationary-state skewness seems to have escaped everyone's attention thus far. The only discussions of skewness in the literature of which we are aware, concern temporal skewness [26–28, 5]. One possible reason is that almost all previous studies involve MC-type simulations. In master equation studies, like ours, the stationary state and its properties are directly accessible (as the eigenvector of the largest eigenvalue of the time evolution operator). Consider the moments of the height distribution, $h_x(t)$, at a certain moment in time, in a finite 1D system with periodic boundary conditions, $h_{x+L}(t) = h_x(t)$,

$$W_n(L, t) = L^{-1} \int \mathrm{d}x \, [h_x(t) - h_{av}(t)]^n. \tag{1.2}$$

The first moment vanishes since it defines the average surface height $h_{av}(t)$. The second moment is the conventional measure for the width of the interface. The third moment characterizes the skewness. In EW-type growth all odd moments vanish because of particle-

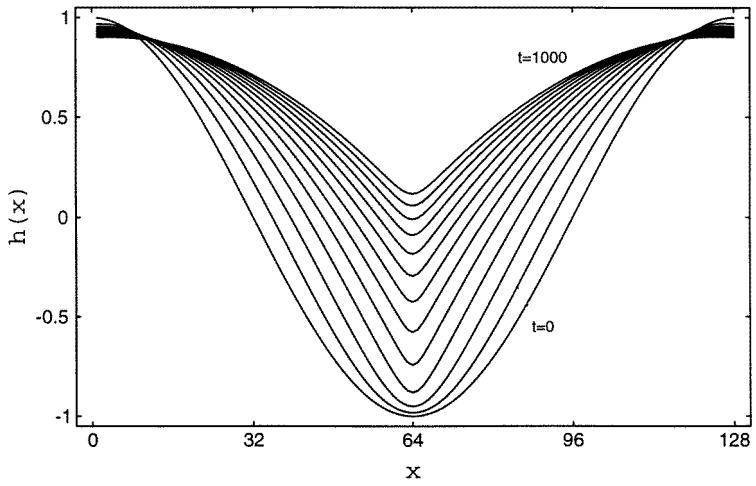

Figure 1. Temporal skewness in the deterministic KPZ equation. Time evolution of a sinusoidal initial state.

hole symmetry at $\lambda = 0$. KPZ-type growth lacks that symmetry. On a local level, skewness means that the curvature at a typical valley-bottom is larger in magnitude than at a typical hill-top (or the other way around); figure 1 illustrates this. It shows the deterministic time evolution of a sinusoid initial state, according to equation (1.1) without noise for positive λ.

It is important to distinguish between stationary-state skewness and temporal skewness. The skewness in figure 1 for the deterministic KPZ equation is a transient phenomenon. This initial state decays to zero, to a stationary state with no skewness whatsoever. Noise pumps sinusoid waves at random frequencies into the surface (from the spatial Fourier transformed perspective) at all time scales. Each mode decays with the same sign for skewness. So in the presence of random noise the stationary state tends to be rough and skewed. This picture is too simplistic. It ignores the nonlinear coupling between the modes. Moreover, the stationary state of equation (1.1) is known exactly in 1D; it is Gaussian. All moments W_n for $n > 2$ vanish [5, 14, 29].

The BCSOS model is one of the simplest microscopic KPZ-type growth rules [30, 31], and it is exactly soluble in 1D [6–9]. Its stationary state is trivial as well. It is the completely disordered state, without any skewness (or any other structure), not only in the thermodynamic limit but also for all finite L. The origin of this is a special symmetry of the time-evolution operator; we review this in section 2. Throughout this paper we will compare our results for the RSOS model with the properties of the BCSOS model.

Our study of the RSOS model illustrates that the simplicity of the stationary states of equation (1.1) and the BCSOS model is accidental. Generically, stationary states are skewed. Skewness can be turned on and off in the RSOS model. In section 3, we derive the exact form of its stationary state in a 4D subspace. This stationary state is simple—skewness and all other correlations between steps in the surface are absent. Steps are placed at random, with not even nearest-neighbour correlations between them. The only parameter is the step density. Familiar special points, such as the Kim–Kosterlitz (KK) model [32], and special lines [33] lie outside this subspace. Their stationary states are non-trivial and stationary state skewness is one aspect of this.

In section 4, we present a mean-field-type derivation of the deterministic part of the KPZ equation for the BCSOS and RSOS models. This derivation provides more insight into the general structure of the RSOS model phase diagram. In particular, it identifies a second order parameter, the step density, and a typical time scale, τ_s. The step density (re-)equilibrates locally at a time scale τ_s. The RSOS model is described by the KPZ equation only at time resolutions less than τ_s. In section 5, we combine these results into a description of the global structure of the RSOS model phase diagram. We also describe how skewness can be introduced and controlled within the Langevin equation, equation (1.1).

Stationary-state skewness raises an important fundamental level-two-type issue. One of the basic premises of renormalization theory is that all short-distance complexities are expressible in terms of irrelevant operators. Stationary-state skewness should be an example of this. It is important to demonstrate this explicitly, since level-two-type understanding of KPZ-type growth is still rudimentary. According to the empirical scaling theory, the growing interface displays scale invariance at large length scales and long times. It is invariant under the transformation: $x' = bx$, $h' = b^{\alpha}h$, and $t' = b^{z}t$. The moments of the height distribution scale as

$$W_n(L,t) = b^{n\alpha} W_n(b^{-1}L, b^{-z}t) \tag{1.3}$$

α is the stationary-state roughness exponent and z the dynamic exponent. Galilean invariance in the Burgers equation representation of equation (1.1) yields the identity $\alpha + z = 2$ [1–5, 14]. In 1D the height variable scales with the exponent $\alpha = \frac{1}{2}$. The width squared of the surface, W_2, diverges linearly with L in the stationary state. The step–step correlation function

$$G_s(r) = \langle \nabla h_{x+r} \nabla h_x \rangle \sim \exp(-r/\xi_s) \tag{1.4}$$

has a finite correlation length, or at least decays fast enough that going up and down along the surface becomes a random walk at large length scales. It is unlikely that KPZ-type models with stationary state skewness belong to a different universality as those without it. The numerical evidence for $z = 1.5$ is very strong, in particular at the KK point. But, are they all described by the same fixed point?

The conventional KPZ fixed point lies at zero skewness. It applies to equation (1.1), to the BCSOS model, and also to the RSOS model inside the non-skewed 4D subspace. This fixed point describes skewed surfaces as well, if we can demonstrate that the skewness crossover operator is irrelevant in the sense of the renormalization theory. This does not mean that aspects such as skewness vanish in the thermodynamic limit. Skewness, W_3, is allowed to diverge with system size, but at a slower rate than its naive exponent, 3α. In section 5 we show numerically, using a master equation finite size scaling analysis, that the skewness in the RSOS model diverges in the thermodynamic limit as $W_3 \sim N^x$, with $x \simeq 0.5$. This confirms that the KPZ fixed point lies at zero skewness. The skewness crossover scaling exponent is equal to $y_{sk} \simeq -1$.

2. The stationary state in the BCSOS model

The stationary state of the BCSOS growth model is simple. It is the completely disordered state. This is a consequence of the symmetry properties of the master equation. First we review these symmetries and then rederive the stationary state by a more complex method, one that can be generalized to the RSOS model in section 3.

Consider a surface built from rectangular shaped bricks (a conventional brick wall turned over 90°). The surface heights at $x = 2n + \frac{1}{2}$ are even integers and those at $x = 2n - \frac{1}{2}$ are

odd. Nearest-neighbour columns differ in height by one unit. Each bond $x = n$ contains a step $S_n = \pm 1$. The growth rule is as follows: Choose one of the columns at random. If this column is at the bottom of a local valley, $S_{n-1} = -1$ and $S_n = +1$, a particle adsorbs with probability p (and nothing happens with probability $1 - p$). If it is at the top of a local hill, $S_{n-1} = +1$ and $S_n = -1$, a particle desorbs with probability q (and nothing happens with probability $1 - q$). If it is part of a local slope, $S_{n-1} = S_n$, nothing happens. This model has been studied extensively in the literature, first by Monte Carlo simulations [30, 3], more recently it was realized it can be solved exactly [6–9].

The master equation

$$|\Psi\rangle_{t+1} = \mathcal{T}|\Psi\rangle_t \tag{2.1}$$

describes the time evolution of the probability distribution $c(\{S_n\})$

$$|\Psi\rangle = \sum_{\{S_n^z\}} c(\{S_n^z\})\ |\{S_n^z\}\rangle. \tag{2.2}$$

The time-evolution operator has the familiar form

$$\mathcal{T} = 1 - N^{-1} \sum_n \mathcal{H}(n, n+1) \tag{2.3}$$

but $\mathcal{H}$ is not Hermitian.

$$\mathcal{H}(n, n+1) = \tfrac{1}{4}\epsilon[1 - S_n^z S_{n+1}^z - 2(S_n^+ S_{n+1}^- + S_n^- S_{n+1}^+) - 2s(S_n^+ S_{n+1}^- - S_n^- S_{n+1}^+)] \tag{2.4}$$

with $s = (p-q)/(p+q)$, and periodic boundary conditions, $S_{n+N}^z = S_n^z$. $\mathcal{T}$ must be applied N times to evolve the system by one unit of time. Without loss of generality we can set $\epsilon = p + q = 1$.

At $s = 0$, $\mathcal{H}$ is identical to the Hamiltonian of the so-called XXZ quantum spin-$\frac{1}{2}$ chain. For any value of s the model is equivalent to the 2D equilibrium six-vertex model in an electric field s. To be more precise, the time-evolution operator is identical to the transfer matrix of the six-vertex model when the sites are being updated sequentially instead of at random. The master equation reduces to equation (2.3) in the time continuum limit. The six-vertex model and equation (2.4) are exactly soluble [7, 8, 34–36]. The master equation follows a special line through the six-vertex model phase diagram, where $\mathcal{T}$ is a stochastic matrix ($\mathcal{T}$ preserves probability). The six-vertex model describes the temperature evolution of equilibrium crystal surfaces [35]. KPZ-type growth maps onto facet-ridge endpoints [17, 34], special points in the phase diagram where the rough (chiral-Luttinger liquid) and faceted phases meet.

The stationary state of equation (2.3) is very simple. It is the disordered state $|D\rangle$ where all coefficients $c(\{S_n\})$ are equal. For non-growing surfaces, with $p = q$, this is obvious. The dynamic rule is then equivalent to a Monte Carlo process in equilibrium statistical mechanics. The coefficients of the stationary state are proportional to the equilibrium Boltzmann weights. $|D\rangle$ is the equilibrium state since this 1D BCSOS model lacks any interactions.

$|D\rangle$ is the stationary state for the growing BCSOS surface as well. Stochastic processes preserve probability. Algebraically, this is expressed by the property that $\langle D|$ is the left eigenvector of the largest eigenvalue $\lambda = 1$ for all stochastic time evolution operators. In general, the adjoint of a time evolution operator is not stochastic, but for the BCSOS model it is. $\mathcal{T}^\dagger$ describes the same growth process, but with the role of particles and holes reversed. The left and right eigenvectors switch roles, and therefore $|D\rangle$ is the right eigenvector for all $s \neq 0$. The stationary state is completely disordered.

The fact that $\mathcal{T}^\dagger$ is stochastic in the entire phase diagram is an accident. It is also an accident that $\mathcal{T}^\dagger$ is identical to the particle-hole transformed time-evolution operator. We

will see this more clearly in section 3 during the discussion of the RSOS model. For that model we will be able to generalize the following method to derive the stationary state.

The BCSOS model can be interpreted as a model for (classical) particles hopping along a chain in an electric field. $S_n^z = 1$ represents an occupied site and $S_n^z = -1$ an empty site. This is also known as the asymmetric exclusion model [10]. Consider the probability $c(\{S_n\})$ to be in microstate $\{S_n\}$. Let the dots in $c(\ldots, S_n^z, S_{n+1}^z, \ldots)$ mean that all spins other than those at sites n and $n+1$ are the same for all states considered and do not change. The action of each individual $\mathcal{H}(n, n+1)$,

$$c(\ldots, +, -, \ldots)_{t+1/N} = (1-q)c(\ldots, +, -, \ldots)_t + pc(\ldots, -, +, \ldots)_t \tag{2.5}$$

leaves $|\Psi\rangle$ invariant when

$$\frac{c(\ldots, +, -, \ldots)}{c(\ldots, -, +, \ldots)} = \frac{p}{q}. \tag{2.6}$$

The electrostatic energy of charged particles hopping along a wire in an electric field $\mathcal{E}$ is equal to $\mathcal{E}\mathcal{K}$, with

$$\mathcal{K} = \sum_n \tfrac{1}{2} n(S_n^z + 1). \tag{2.7}$$

A conventional MC simulation rule for such a system obeys detailed balance, in which the transition probabilities between states a and b are related as

$$\frac{P_{a\to b}}{P_{b\to a}} = \exp(-\mathcal{E}(\mathcal{K}_b - \mathcal{K}_a)/k_B T). \tag{2.8}$$

This is precisely the content of equation (2.6). The stationary state is the 'atmospheric law' density distribution:

$$\rho(n) = \exp(-\mathcal{E}n/k_B T) \tag{2.9}$$

and $p/q = \exp(\mathcal{E}/k_B T)$. This is not the solution we are looking for; it applies to the wrong boundary conditions. Equation (2.9) is valid for open boundary conditions. In our case the particles keep running around in a circle and no density profile can build up to stop this flow. The detailed balance approach is apparently too restrictive. We must attack the problem less locally.

The following property will prove essential. The number of hill-tops and valley-bottoms is the same in every configuration [2]. This is easily established by drawing a typical configuration with periodic boundary conditions and a specific average slope. Algebraically it can be shown as follows. Consider the four nearest-neighbour step–step densities in each configuration: d_{+-}, d_{-+}, d_{++}, and d_{--}. They represent respectively, the density of hill-tops, valley-bottoms, and up and down slopes. The density of up-spins, d_+, is related by normalization as

$$d_+ = d_{++} + d_{-+} = d_{++} + d_{+-}. \tag{2.10}$$

This implies that $d_{+-} = d_{-+}$.

Let us test the following assumption: The probability distribution $c(\{S_n\})$ depends only on the total number of particles, $\mathcal{N}$, and the total electrostatic energy $\mathcal{K}$. The average slope of the surface is fixed by the boundary conditions. Therefore $\mathcal{N}$ is a constant of motion and can be ignored. This leaves only the dependence on the electrostatic energy, $c(\{S_n\}) = c(\mathcal{K})$. This probability distribution time evolves according to equation (2.3) as

$$\frac{\partial c(\mathcal{K})}{\partial t} = -(pd_{-+} + qd_{+-})c(\mathcal{K}) + pd_{+-}c(\mathcal{K}+1) + qd_{-+}c(\mathcal{K}-1). \tag{2.11}$$

The first term on the right-hand side represents the contributions when nothing happens between $t \to t + 1/N$. The second and third terms represent events where one particle adsorbs or evaporates. Adsorption destroys one valley-bottom and creates one hill-top. Evaporation does the opposite. The adsorption probability is proportional to d_{+-} since this could have happened to any of the hill-tops in existence at time $t + 1/N$. It is also proportional to $C(\mathcal{K}+1)$, because in the electric-field interpretation one particle hops in the direction of the electric field.

Our assumption for the stationary state is correct if the right-hand side of equation (2.11) vanishes for all $d_{i,j}$. This seems to imply two conditions, but there is only one, since d_{-+} can be eliminated using the identity $d_{-+} = d_{+-}$. The right-hand side vanishes for all configurations if

$$p + q = p/w + qw \tag{2.12}$$

with $w = c(\mathcal{K})/c(\mathcal{K}+1)$. This yields two stationary state solutions: the detailed balance solution with $w = p/q$, and the stationary growing state solution $|D\rangle$ with $w = 1$, the state we are looking for.

Without the global identity $d_{-+} = d_{+-}$, equation (2.10) would yield two conditions, one for d_{-+} and one for d_{+-}. These reproduce only the detailed balance solution. The stationary growth solution with $w = 1$ is the result of the non-local property that there are just as many valley-bottoms as hill-tops. Every hill-top (valley bottom) could have been created during $t \to t+1/N$ with probability $p(q)$ and every valley-bottom (hill-top) can be destroyed with the same probability. Since $d_{-+} = d_{+-}$, the right-hand side of equation (2.11) vanishes when the coefficients $c(K)$ are independent of $\mathcal{K}$.

3. The stationary state in the RSOS model

The RSOS growth model describes the growth of simple cubic surfaces in which only monatomic steps are allowed. Nearest-neighbour columns can differ by only $dh = S_n^z = 0, \pm 1$. The master equation is more complex than equation (2.3) for the BCSOS model, since more types of local configurations are possible. Each has its own transition probabilities. After choosing one of the columns at random, one particle can be evaporated or deposited at $x = n + \frac{1}{2}$ with a probability that depends on the height differences S_n^z and S_{n+1}^z. The time-evolution operator is again in the form of equation (2.3) with

$$\begin{aligned}\mathcal{H} = \sum_{n=1,N} \{&[p_h(1 - S_n^+ S_{n+1}^-) + q_v(1 - S_n^- S_{n+1}^+)]\delta(0)_n\delta(0)_{n+1} \\ &+q_h(1 - S_n^- S_{n+1}^+)\delta(+)_n\delta(-)_{n+1} + p_v(1 - S_n^+ S_{n+1}^-)\delta(-)_n\delta(+)_{n+1} \\ &+p_s(1 - S_n^+ S_{n+1}^-)[\delta(0)_n\delta(+)_{n+1} + \delta(-)_n\delta(0)_{n+1}] \\ &+q_s(1 - S_n^- S_{n+1}^+)[\delta(+)_n\delta(0)_{n+1} + \delta(0)_n\delta(-)_{n+1}]\}\end{aligned} \tag{3.1}$$

and the following definitions: $\delta(0) = (1 + S^z)(1 - S^z)$, $\delta(+) = \frac{1}{2}S^z(S^z + 1)$, and $\delta(-) = \frac{1}{2}S^z(S^z - 1)$. The raising and lowering operators S^+ and S^- are normalized such that $S^z = S^+S^- - S^-S^+$ and $S^+S^- + S^-S^+ = 2 - (S^z)^2$. This time-evolution operator has six parameters. They have the following interpretations: p_h is the adsorption probability and q_v is the evaporation probability when the surface is locally flat; q_h is the evaporation probability at a local hill-top; p_v is the adsorption probability into a local valley-bottom; p_s is the adsorption probability and q_s the evaporation probability at a step. These transition rates are demonstrated in figure 2.

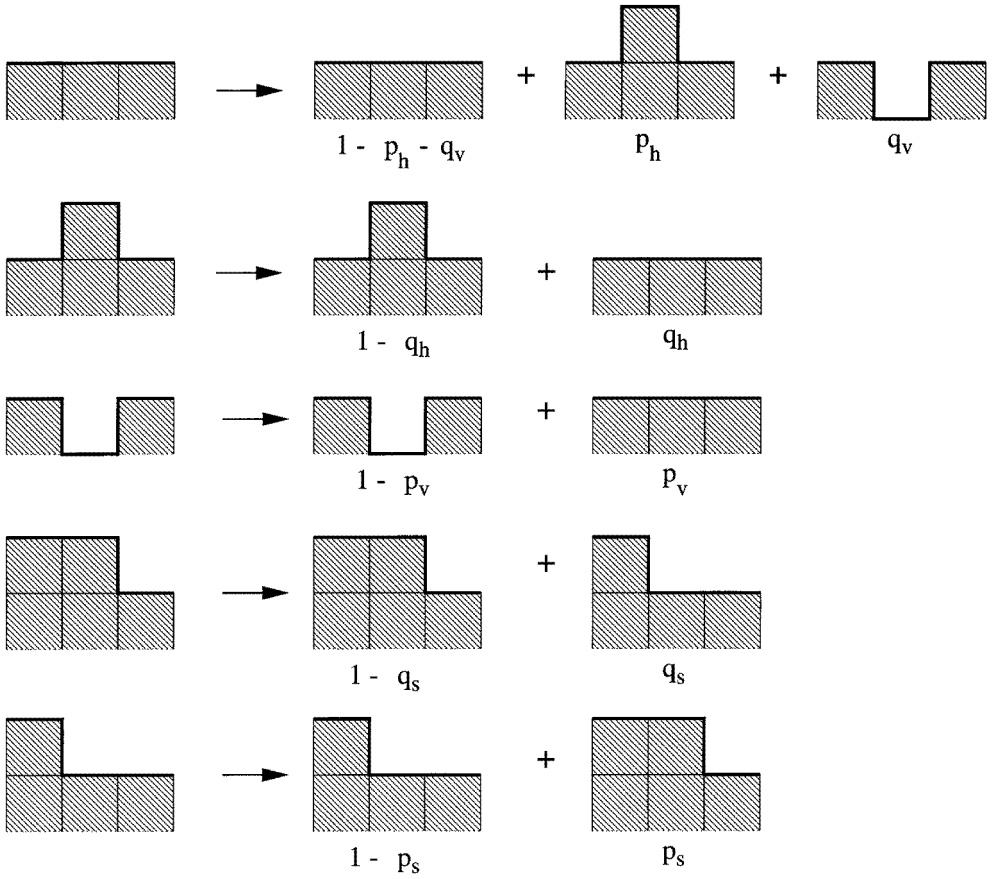

Figure 2. RSOS growth parameters. Sketch of representative surface configurations and their transition rates.

$\mathcal{H}$ is not Hermitian, but resembles the Hamiltonian of a quantum spin-1 chain.

$$\mathcal{H}_D(n,n+1) = c_s + (s_m - 2c_s)(S_n^z)^2 - \tfrac{1}{4}a_s S_n^z S_{n+1}^z + (c_s - s_m + \tfrac{1}{4}a_s)(S_n^z S_{n+1}^z)^2 + (\tfrac{1}{2}s_d - \tfrac{1}{4}h_g)S_n^z S_{n+1}^z (S_{n+1}^z - S_n^z) \tag{3.2a}$$

the diagonal part, and

$$\mathcal{H}_{OD}(n,n+1) = -\mathcal{F}_+ S_n^+ S_{n+1}^- - \mathcal{F}_- S_n^- S_{n+1}^+ \tag{3.2b}$$

the off-diagonal part, with

$$\begin{aligned}\mathcal{F}^+ &= (p_h - p_s)(S_n^z S_{n+1}^z)^2 + (p_v - p_s)[1-(S_n^z)^2][1-(S_{n+1}^z)^2] + p_s \\ \mathcal{F}^- &= (q_v - q_s)(S_n^z S_{n+1}^z)^2 + (q_h - q_s)[1-(S_n^z)^2][1-(S_{n+1}^z)^2] + q_s.\end{aligned} \tag{3.2c}$$

In $\mathcal{H}_D$ we introduced a second notation for the six parameters: $c_s = p_h + q_v$ is the step creation probability; $a_s = p_v + q_h$ is the step annihilation probability; $s_m = p_s + q_s$ is the step mobility; $f_g = p_h - q_v$ is the growth probability at flat surface areas; $h_g = p_v - q_h$ is the growth probability at hill-tops and valleys; and $s_d = p_s - q_s$ is the growth probability at steps (step drift).

The phase diagram is only 5D because rescaling all six parameters by a common factor redefines the unit of time. Some familiar models are contained in this master equation

as special points or lines [32, 33]. For example, the KK model [32] corresponds to $q_v = q_h = q_s = 0$ (no evaporation) and $p_v = p_h = p_s$.

Recall (see section 2) that the stationary state of the BCSOS model, equation (2.4), is trivial. It is the completely disordered state $|D\rangle$, because the adjoint of its time-evolution operator it also stochastic. It describes the time evolution of the same surface in the particle-hole transformed representation. The adjoint of equation (3.1) is only stochastic in a three-dimensional (3D) subspace of the phase diagram. The diagonal parts of $\mathcal{T}^\dagger$ and $\mathcal{T}$ are identical, but the coefficients in the off-diagonal part switch position,

$$\mathcal{T}(c_s, a_s, s_m; f_g, h_g, s_d)^\dagger = \mathcal{T}_D(c_s, a_s, s_m; f_g, h_g, s_d) + \mathcal{T}_{OD}(a_s, c_s, s_m; -h_g, -f_g, -s_d). \tag{3.3}$$

The parameter space in which $\mathcal{T}^\dagger$ preserves probability is larger than the 2D self-adjoint subspace in which the surface does not grow ($p_h = q_h$, $p_v = q_v$, and $p_s = q_s$). $\mathcal{T}^\dagger$ is stochastic whenever the diagonal part $\mathcal{T}_D$ in equation (3.3), is invariant under the transformation $c_s \leftrightarrow a_s$, $h_g \leftrightarrow -f_g$, and $s_d \to -s_d$ (the way the parameters switch in the off-diagonal part). This is true for

$$c_s = a_s \qquad h_g - 2s_d = -f_g + 2s_d. \tag{3.4}$$

Inside this 3D subspace the adjoint of the time-evolution operator is stochastic and describes surfaces in which the probabilities are interchanged as

$$\mathcal{T}(c_s, a_s, s_m; f_g, h_g, s_d)^\dagger = \mathcal{T}(a_s, c_s, s_m; -h_g, -f_g, -s_d). \tag{3.5}$$

In the BCSOS model, $\mathcal{T}^\dagger$ is identical to the particle-hole transformed dynamics. This is not true any more in the RSOS model; the particle-hole transformed time-evolution operator

$$\widehat{\mathcal{O}}_{PH}\mathcal{T}(c_s, a_s, s_m; f_g, h_g, s_d) = \mathcal{T}(c_s, a_s, s_m; -f_g, -h_g, -s_d) \tag{3.6}$$

is different to the one in equation (3.5).

Next, we generalize this stationary state to a 4D subspace. The BCSOS model master equation describes a lattice gas of non-interacting charged particles in an electric field. The RSOS master equation is a particle-hole generalization of this. The $S_n^z = \pm 1$ states represent sites occupied by particles with electric charge ± 1. $S_n^z = 0$ states represent empty sites. The energy of such a lattice gas is of the form

$$E = \sum_n [\mu (S_n^z)^2 + \mathcal{E} n S_n^z]. \tag{3.7}$$

The chemical potential μ is needed, because the dynamic rule does not conserve the total number of particles (the number of steps in the surface). The chemical potential for the total charge can be omitted because electric charge is conserved. We are looking for a stationary state of the form $c(\{S_n^z\}) = c(\mathcal{N}_s, \mathcal{K})$ in which the coefficients only depend on the number of particles, $\mathcal{N}_s$ and the total electrostatic energy, $\mathcal{K} = \sum_n n S_n^z$,

$$c(\{S_n^z\}) \sim \exp[-\mu \mathcal{N}_s - \mathcal{E}\mathcal{K}] = z^{\frac{1}{2}N_s} w^{\mathcal{K}}. \tag{3.8}$$

Let us try the conventional detailed balance approach for Monte Carlo simulations. Require that the action of each individual $\mathcal{H}_{n,n+1}$ leaves $|\Psi\rangle$ invariant. This yields the conditions

$$\begin{aligned} \frac{q_h}{p_h} &= \frac{c(\ldots, 0, 0, \ldots)}{c(\ldots, +, -, \ldots)} \\ \frac{p_v}{q_v} &= \frac{c(\ldots, 0, 0, \ldots)}{c(\ldots, -, +, \ldots)} \\ \frac{p_s}{q_s} &= \frac{c(\ldots, +, 0, \ldots)}{c(\ldots, 0, +, \ldots)} = \frac{c(\ldots, 0, -, \ldots)}{c(\ldots, -, 0, \ldots)} \end{aligned} \tag{3.9}$$

(using the same type of notation as in equation (2.5)). They are satisfied by the stationary state in the form of equation (3.8) when

$$\begin{aligned} p_h/q_h &= \exp[(\mathcal{E}-2\mu)/k_BT] \\ p_v/q_v &= \exp[(\mathcal{E}+2\mu)/k_BT] \\ p_s/q_s &= \exp[\mathcal{E}/k_BT]. \end{aligned} \tag{3.10}$$

This defines a 4D subspace inside the 5D phase diagram, since each set of values $\mathcal{E}$ and μ leaves two ratios unspecified, p_h/p_v and p_h/p_s.

Equation (3.9) is not the stationary state we are interested in. It applies to the wrong type of boundary condition. Let us apply the same non-local approach as in section 2. Assume that the stationary state is in the form of equation (3.8) and express the time evolution in terms of the nine nearest-neighbour step–step densities d_{00}, $d_{+-},\ldots,$ d_{++}. The equations of motion read:

$$\begin{aligned} \frac{\partial c(\mathcal{N}_s,\mathcal{K})}{\partial t} &= -[(p_h+q_v)d_{00}+p_vd_{-+}+q_hd_{+-}]c(\mathcal{N}_s,\mathcal{K}) \\ &\quad +p_hd_{+-}c(\mathcal{N}_s-2,\mathcal{K}+1)+q_vd_{-+}c(\mathcal{N}_s-2,\mathcal{K}-1) \\ &\quad +q_hd_{00}c(\mathcal{N}_s+2,\mathcal{K}-1)+p_vd_{00}c(\mathcal{N}_s+2,\mathcal{K}+1) \\ &\quad +q_s(d_{0+}+d_{-0})c(\mathcal{N}_s,\mathcal{K}-1)+p_s(d_{+0}+d_{0-})c(\mathcal{N}_s,\mathcal{K}+1) \\ &\quad -[p_s(d_{0+}+d_{-0})+q_s(d_{+0}+d_{0-})]c(\mathcal{N}_s,\mathcal{K}). \end{aligned} \tag{3.11}$$

This is the generalization of equation (2.11). The stationary state is in the form of equation (3.8), if the right-hand side vanishes for all $d_{i,j}$. Naively this yields five equations with five unknowns. However, the $d_{i,j}$ in equation (3.11) are not independent. The analogues of equation (2.10) are

$$\begin{aligned} d_+ &= d_{++}+d_{+0}+d_{+-} = d_{++}+d_{0+}+d_{-+} \\ d_- &= d_{--}+d_{0-}+d_{+-} = d_{--}+d_{-0}+d_{-+} \end{aligned} \tag{3.12}$$

with d_+ and d_- the density of up- and down-steps, and $d_0+d_++d_-=1$. Adding and subtracting these two equations yields two identities:

$$\begin{aligned} 2d_{+-}+d_{+0}+d_{0-} &= 2d_{-+}+d_{0+}+d_{-0} \\ d_{+0}-d_{0-} &= d_{0+}-d_{-0}. \end{aligned} \tag{3.13}$$

After employing the first one, only d_{00}, d_{+-}, d_{-+}, and $d_{+0}+d_{0-}$ remain in equation (3.11). Equation (3.8) is the correct stationary state if the following four conditions are satisfied:

$$\begin{aligned} (p_h+q_v) &= q_hzw^{-1}+p_vzw \\ q_h &= p_hz^{-1}w-2(p_s-q_sw^{-1}) \\ p_v &= q_v(zw)^{-1}+2(p_s-q_sw^{-1}) \\ 0 &= (p_s-q_sw^{-1})(1-w). \end{aligned} \tag{3.14}$$

The last equation indicates that there are two solutions: one with $q_s/p_s=w$ and one with $w=1$. The first solution, with $w=q_s/p_s$, reproduces the atmospheric law-type charge profile (3.9). The second solution, with $w=1$, applies to periodic boundary conditions and is the one we are looking for. For $w=1$, the last condition in (3.14) is satisfied without any constraints on p_s and q_s. Only two of the three remaining conditions are independent. One sets the step density and the other specifies one condition between the five growth parameters. The stationary state is of the form

$$c(\{S_n^z\}) = c(\mathcal{N}_s) \sim \exp[-\mu\mathcal{N}_s] = (\sqrt{z})^{\mathcal{N}_s} \tag{3.15a}$$

inside the 4D subspace defined as

$$z = \frac{p_h + q_v}{q_h + p_v}$$
$$0 = z(p_v - q_h) + (p_h - q_v) - 4z(p_s - q_s) \tag{3.15b}$$

the latter is equivalent to

$$z = c_s/a_s$$
$$4s_d z = zh_g + f_g. \tag{3.15c}$$

Both solutions of w are valid in a subspace of co-dimension one of the 5D parameter space, but these subspaces do not coincide, nor do the equations for the step-fugacity z.

In the stationary state (3.15) the step density is equal to

$$s = \langle D|(S_n^z)^2|\Psi\rangle = \frac{2\sqrt{z}}{1+2\sqrt{z}} \tag{3.16}$$

for zero surface tilt, $v = \langle D|S_n^z|S\rangle = 0$. Notice that for $z = 1$ we recover the disordered state solution, equation (3.4).

The stationary growth rate, r_g, is equal to

$$\begin{aligned} r_g &= \langle D|(\mathcal{F}_+ S_n^+ S_{n+1}^- - \mathcal{F}_- S_n^- S_{n+1}^+)|\Psi\rangle \\ &= f_g(1-s)^2 + \tfrac{1}{4}h_g s^2 + s_d s(1-s) \end{aligned} \tag{3.17}$$

which simplifies with equation (3.15*c*) and (3.16) to

$$r_g = s_d s. \tag{3.18}$$

The height–height correlation function diverges linearly because the spins are uncorrelated:

$$\langle D|(h_{n+\frac{1}{2}} - h_{n+m+\frac{1}{2}})^2|\Psi\rangle = \langle D|\sum_{n<i\leqslant n+m}(S_i^z)^2|\Psi\rangle = ms. \tag{3.19}$$

This confirms that the surface roughness critical exponent is equal to $\alpha = \frac{1}{2}$.

It is amazing that the stationary state is this simple in such a large fraction of the phase diagram. The step–step correlation function, equation (1.4), has a sharp cut-off, to such an extent that steps are completely disordered. Outside this 4D subspace the stationary state becomes more complex. Equation (1.4) does not have a sharp cut-off any more. We checked this numerically. Attempts to extend the above derivation to a form in which the $c(\{S_n\})$ are functions only of the nearest-neighbour step–step correlations are doomed. The number of equations increases rapidly and they cannot be satisfied inside the 5D parameter space.

4. Mean-field theory and Langevin equations

In this section we apply the mean-field theory to the BCSOS and RSOS model. The mean-field equations of motion for the order parameters reproduce the deterministic part of the KPZ equation. This is similar to equilibrium critical phenomena, where mean-field theory for the Ising model reproduces only the non-fluctuating part of the ϕ^4-theory. The aim is to improve the understanding of the structure of the 5D RSOS model phase diagram; in particular, to relate qualitatively the parameters of the RSOS model to those in the KPZ equation.

Consider the BCSOS master equation, equation (2.4). The equation of motion

$$\frac{\partial v}{\partial t} = v_n(t+1) - v_n(t) = \langle D|[S_n^z, \mathcal{T}]|\Psi\rangle_t \tag{4.1}$$

for the local slope of the surface

$$v_n(t) = \langle D|S_n^z|\Psi\rangle_t \tag{4.2}$$

does not close because of the presence of nearest-neighbour step–step correlations on the right-hand side. Define

$$d(a,b)_x = \langle D|\delta(S_n^z - a)\delta(S_{n+1}^z - b)|\Psi\rangle \tag{4.3}$$

with $a, b = \pm 1$ and $x = n + \frac{1}{2}$, and replace the slope variable by the surface height using the relation $v_n = h_{x+1} - h_x$. Equation (4.1) yields,

$$\frac{\partial h}{\partial t} = pd(+,-)_x - qd(-,+)_x. \tag{4.4}$$

This is the top of a BBGKY hierarchy of equations of motion. In the lowest-order approximation all step–step correlation are neglected. The $d(a,b)_x$ are assumed to factorize. The equation of motion for the average local slope $v_n(t)$ then closes.

$$\frac{\partial}{\partial t}h_x = \tfrac{1}{4}p(1 - v_n)(1 + v_{n+1}) - \tfrac{1}{4}q(1 + v_n)(1 - v_{n+1}). \tag{4.5}$$

A faster route to exactly the same equation is to apply the mean-field approximation directly to the master equation. The adsorption and evaporation transition rates at site x are assumed to depend on the expectation values of the step density v_n instead of their actual values S_n^z. Equation (4.5) is identical to the deterministic part of the KPZ equation, equation (1.1), in the spatial continuum limit

$$\frac{\partial h}{\partial t} = \frac{1}{4}(p-q) + \frac{1}{4}(p+q)\frac{\partial^2 h}{\partial x^2} - \frac{1}{4}(p-q)\left(\frac{\partial h}{\partial x}\right)^2 \tag{4.6}$$

λ is negative and reduces the growth rate at slopes. This reflects that in the BCSOS model sloped surface sections, with $S_n = S_{n+1}$, are inactive.

We will now apply the same type of approximation to the RSOS model. The first observation is that the RSOS model contains two point-like expectation values instead of one: the local step density, s_n, and the local slope of the surface, v_n,

$$\begin{aligned} s_n(t) &= \langle D|(S_n^z)^2|\Psi\rangle_t \\ v_n(t) &= \langle D|S_n^z|\Psi\rangle_t. \end{aligned} \tag{4.7}$$

The mean-field theory involves two order parameters instead of one. The equations of motion for these two order parameters,

$$\frac{\partial s}{\partial t} = s_i(t+1) - s_i(t) = \langle D|[(S_i^z)^2, \mathcal{T}]|\Psi\rangle_t \tag{4.8a}$$

$$\frac{\partial v}{\partial t} = v_i(t+1) - v_i(t) = \langle D|S_i^z, \mathcal{T}]|\Psi\rangle_t \tag{4.8b}$$

are again the top of a BBGKY-type hierarchy of coupled differential equations:

$$\begin{aligned} \frac{\partial s_{x+\frac{1}{2}}}{\partial t} &= (p_h + q_v)[d(0,0)_x + d(0,0)_{x+1}] \\ &\quad - p_v[d(-,+)_x + d(-,+)_{x+1}] - q_h[d(+,-)_x + d(+,-)_{x+1}] \\ &\quad + p_s[d(-,0)_x - d(0,+)_x - d(-,0)_{x+1} + d(0,+)_{x+1}] \\ &\quad + q_s[d(+,0)_x - d(0,-)_x - d(+,0)_{x+1} + d(0,-)_{x+1}] \end{aligned} \tag{4.9a}$$

$$\begin{aligned} \frac{\partial h_x}{\partial t} &= (p_h - q_v)d(0,0)_x + p_v d(-,+)_x - q_h d(+,-)_x + p_s[d(-,0)_x + d(0,+)_x] \\ &\quad - q_s[d(+,0)_x + d(0,-)_x]. \end{aligned} \tag{4.9b}$$

In the mean-field theory the nearest-neighbour step–step correlations factorize. The resulting equations are not very transparent. Therefore, we first consider the special case where the surface is spatially uniform, such that $s = s_n$ and $h = h_x$ do not depend on position. The mean-field approximation for equation (4.9) then reads

$$\frac{\partial s}{\partial t} = 2c_s(1-s)^2 - \tfrac{1}{2}a_s(s-v)(s+v) \tag{4.10a}$$

$$\frac{\partial h_n}{\partial t} = f_g(1-s)^2 + \tfrac{1}{4}h_g(s^2-v^2) + s_d(1-s)s. \tag{4.10b}$$

The equation for the step density contains a 'mass' term, which defines a characteristic time τ_s. At zero net tilt, $v = 0$, the step density reaches its stationary value

$$s_0 = \left[1 + \frac{1}{2}\sqrt{\left(\frac{a_s}{c_s}\right)}\right]^{-1} \tag{4.11a}$$

after a characteristic time

$$\tau_s^{-1} = 2\sqrt{a_s c_s}. \tag{4.11b}$$

At time scales larger than τ_s the surface grows at a constant average rate

$$r_0 = f_g(1-s_0)^2 + \tfrac{1}{4}h_g s_0^2 + s_d(1-s_0)s_0. \tag{4.11c}$$

This means that, although there are two order parameters, only one of them, the local slope v_n, fluctuates at time scales larger than τ_s. The full mean-field equation for the step density reads:

$$\begin{aligned}\frac{\partial s}{\partial t} = {} & 2c_s(1-s)^2 - \frac{1}{2}a_s(s^2-v^2) + s_d\frac{\partial}{\partial x}[(1-s)v] + \frac{1}{2}h_g\left(v\frac{\partial s}{\partial x} - s\frac{\partial v}{\partial x}\right) \\ & + \left(\frac{1}{2}s_m - c_s\right)\frac{\partial^2 s}{\partial x^2} + \frac{1}{4}a_s v\frac{\partial^2 v}{\partial x^2} + \left(c_s - \frac{1}{4}a_s\right)s\frac{\partial^2 s}{\partial x^2}\end{aligned} \tag{4.12}$$

in which all derivatives are discrete. We will now define Δ as the deviation from the stationary value, $s = s_0 + \Delta$. At larger time scales the step density does not behave as an independent dynamic variable. Δ follows local fluctuations in the slope of the surface such that the right-hand side of equation (4.12) remains equal to zero:

$$m_s\Delta = \delta\left(\frac{\partial h}{\partial x}\right)^2 + \epsilon\frac{\partial^2 h}{\partial x^2} + \cdots \tag{4.13}$$

with $\delta = \frac{1}{2}a_s$ and $\epsilon = s_d(1-s_0) - \frac{1}{2}h_g s_0$. The full equation of motion for h_x is complex. It contains many terms, involving the step density, derivatives of the surface slope, and combinations of these and their derivatives. At time scales larger than τ_s, the step density is not an independent variable, and can be eliminated using equation (4.13). The equation for $\mathrm{d}h/\mathrm{d}t$ then reduces to the deterministic part of the KPZ equation, equation (1.1), with

$$\begin{aligned}\lambda &\simeq \tfrac{1}{2}a_s\tau_s[h_g s_0 - 4f_g(1-s_0) + 2s_d(1-2s_0)] - \tfrac{1}{2}h_g \\ \nu &\simeq \tfrac{1}{2}s_m(1-s_0) + \tfrac{1}{4}a_s s_0\end{aligned} \tag{4.14}$$

and many higher-order terms as well. The above two derivations illustrate that the BCSOS and RSOS growth models belong to the KPZ universality class. That is hardly a surprise however. The significance is the identification of an additional order parameter and of the time scale τ_s. At short time scales the step density behaves as an independent dynamic variable and the KPZ equation description is incomplete. Fortunately τ_s is typically very short, of the order of only a few time steps.

5. Skewness in the RSOS model

The phase diagram of the RSOS model seems too large to visualize, but the results of the previous sections put some order into it. The uncorrelated stationary state (in the subspace defined by equation (3.15)) and the mean-field theory define a set of characteristic densities and time scales. The six parameters in equation (3.2) fall into two groups: c_s, a_s, and s_m, are equilibration-type parameters; f_g, h_g, and s_d, are growth-type parameters.

The ratio between the step creation and annihilation probabilities, $z = c_s/a_s$, controls the steps density, s. Their product $\tau_s^{-1} = 2\sqrt{a_s c_s}$ controls the time scale at which the step density equilibrates. The step mobility s_m is a suitable unit of time. (We set $s_m = 1$.) KPZ-type scaling is realized at time scales larger than τ_s and length scales larger than the inverse of the step density. This leaves us with an effective 3D phase diagram, characterized by the growth parameters, f_g, h_g, and s_d. The linear combination

$$r_g = f_g(1 - s^2) + \tfrac{1}{4}h_g s^2 + s_d s(1 - s) \tag{5.1}$$

with s defined in equation (3.15), controls the growth rate. The linear combination

$$u_{sk} = z h_g + f_g - 4 z s_d \tag{5.2}$$

controls the skewness of the stationary state, see equation (3.15). In the disordered subspace, u_{sk} is identically zero as is the skewness. Choosing the RSOS parameters so that $u_{sk} \neq 0$ takes us away from this plane, and, correspondingly, the skewness is found to be non-zero. In the zero skewness plane, $u_{sk} = 0$, r_g is exactly equal to the growth rate and s is exactly equal to the step density. The third independent combination of f_g, h_g, and s_d controls the strength of the nonlinear term of the KPZ equation. Equation (4.14) is an approximation for λ.

The phase diagram of the RSOS model is effectively only 3D, but is still larger than the one for the KPZ equation, equation (1.1). The RSOS model allows us to control skewness. Stationary skewness can be turned on in the KPZ equation as well. The stationary state of equation (1.1) is known exactly in 1D. It is a Gaussian distribution without any skewness [5, 14, 29]; this is an accident. Additional operators in the equation should introduce stationary state skewness. The following λ_2 type term is an example of this

$$\frac{\mathrm{d}h}{\mathrm{d}t} = r_g + \nu\nabla^2 h + \tfrac{1}{2}\lambda(\nabla h)^2 + \lambda_2(\nabla^2 h)^2 + \eta. \tag{5.3}$$

We performed some (qualitative) Monte Carlo runs at $\lambda_2 \neq 0$ [37] which confirm that the stationary state is skewed. The BCSOS growth model and the $u_{sk} = 0$ RSOS model represent special cuts through the extended KPZ equation, where parameters such as λ_2 take special values such that the stationary state lacks skewness. On a qualitative level the role of λ_2 can be understood as follows. In a flat surface, the growth rate in the RSOS model is determined by f_g, and in the KPZ equation by r_g. In a sloped surface, the growth rate in the RSOS model is determined by $f_g(1 - s_0) + s_d s_0$, and in the KPZ equation by $r_g + \lambda$. In a hilly surface without skewness with densely packed steps, the growth rate in the RSOS model is determined by h_g, and in the KPZ equation, equation (5.3), by $r_g + \lambda_2$.

We hoped that the mean-field theory (section 4) would provide a meaningful estimate of the $u_{sk} = 0$ space in the generalized KPZ equation. Unfortunately, too many operators seem to be involved. For example, the analysis of section 4 gives a non-zero value for λ_2 in the BCSOS model. Apparently that value for λ_2 is compensated by significant contributions of other operators such that stationary state skewness remains absent.

The reverse route might be more promising. The major difference between equation (5.3) and microscopic models such as the BCSOS and RSOS model is the manner in which the noise couples to the local surface structure. In the Langevin equation they are 'additive'. The probability distribution obeys a Fokker–Planck equation [5, 38]. For example, in discretized time it is allowed to visualize the time evolution as deterministic between $t \to t + \frac{1}{2}$ and purely stochastic (ballistic deposition-like) between $t + \frac{1}{2} \to t + 1$.

$$\begin{aligned} h(t + \tfrac{1}{2}) &= h(t) + r_g + \nu\nabla^2 h + \tfrac{1}{2}\lambda(\nabla h)^2 + \lambda_2(\nabla^2 h)^2 \\ h(t + 1) &= h(t + \tfrac{1}{2}) + \eta. \end{aligned} \tag{5.4}$$

In the BCSOS and RSOS model the noise couples much more directly to the local surface configuration. Correlation effects between steps are more pronounced.

The KPZ fixed point must lie inside the non-skewness subspace. As a rule, fixed points lie in subspaces where the short distance properties mimic as much as possible the proper long distance scaling properties. For example, the fixed point of the Ising model on a square lattice lies in the subspace where the next nearest-neighbour interactions are of the same order of magnitude as the nearest-neighbour interactions, such that the correlation functions are rotational invariant not only at large distances, but also at short distances. The cubic anisotropy is an irrelevant operator and needs to be turned off at the fixed point.

The following numerical results demonstrate that stationary state skewness is indeed irrelevant in KPZ-type growth. We determine the largest eigenvalues and eigenvectors of the time evolution operator for system sizes $4 \leqslant L \leqslant 14$. These values are exact, and allow a detailed finite size scaling (FSS) analysis. This method is identical to conventional transfer matrix FSS calculations in equilibrium phase transitions, and works equally well for master equations [17]. We applied this method to several points in the phase diagram, but we present only our results for the KK model [32]. This is a typical point outside the $u_{sk} = 0$ subspace.

On a local level skewness implies an imbalance between sharp hill-tops and valleys-

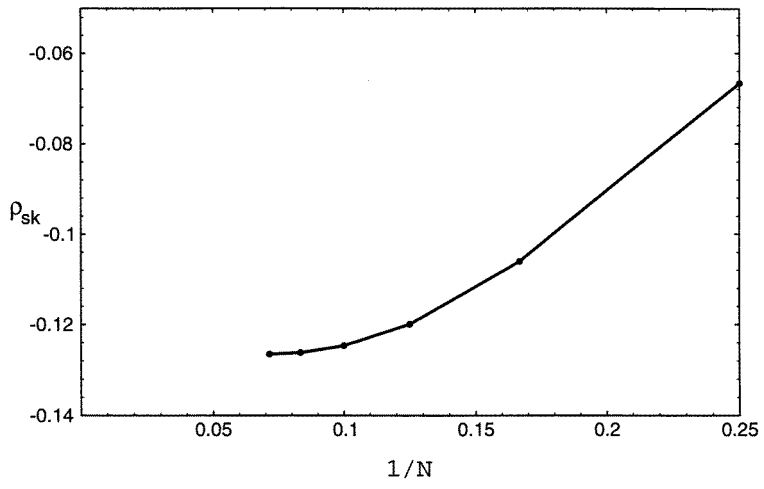

Figure 3. Finite size scaling of the local skewness order parameter, ρ_{sk}, equation (5.5), at the KK point. ρ_{sk} represents the density difference between sharp hill-tops and sharp valley-bottoms.

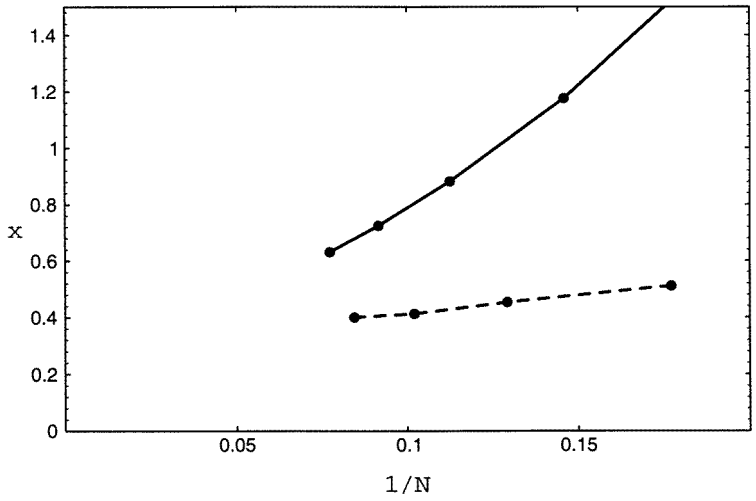

Figure 4. The scaling exponent x of the third moment, $W_3 \simeq AN^x$, at the KK point. The drawn line represents finite size scaling estimates $x(N)$ from the exact values of $W_3(N)$ at $N-1$ and $N+1$ for $N = 5, 7, \ldots 13$. The broken curve shows estimates for x defined as $x(N) \simeq x + B/N^2$ at successive values of N.

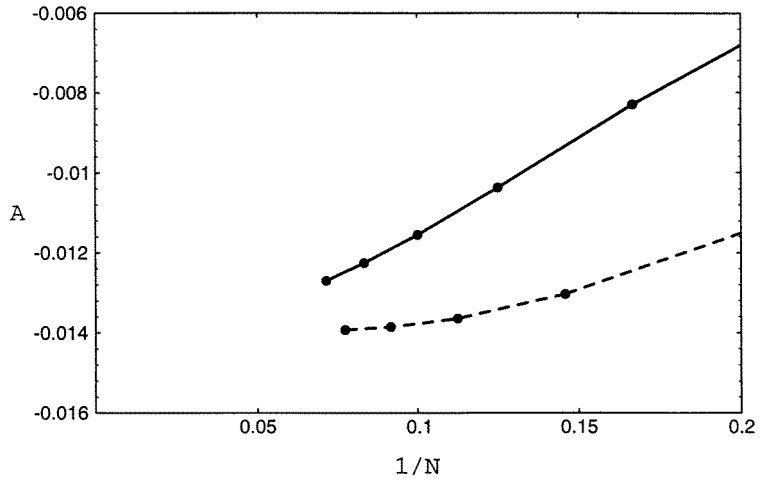

Figure 5. Finite size scaling estimates of the amplitude A of the third moment, $W_3 \simeq AN^x$, at the KK point, assuming that $x = 0.4$. The broken curve represents the same type of extrapolation as in figure 4.

bottoms. The quantity

$$\rho_{sk} = \frac{d(+,-) - d(-,+)}{d(+,-) + d(-,+)} \tag{5.5}$$

measures the density difference between sharp hill-tops and sharp valley-bottoms. ρ_{sk} converges to a non-zero value at the KK point, as shown in figure 3. This illustrates the presence of skewness in the stationary state at short distances. The long distance probe for skewness is the third moment, W_3, of the height distribution function, see equation (1.2).

The scaling behaviour of W_3 tells us whether stationary state skewness is present or

absent at the KPZ fixed point. It should diverge as $W_3 \sim L^{3\alpha}$ if present. Otherwise, we must incorporate into equation (1.3) the crossover scaling in the skewness direction u_{sk},

$$W_n(L, t, u_{sk}) = b^{n\alpha} W_n(b^{-1}L, b^{-z}t, b^{y_{sk}}u_{sk}) \tag{5.6}$$

and expand this in small u_{sk},

$$W_n(L, \infty, u_{sk}) \simeq L^{n\alpha}[W(1, \infty, 0) + L^{y_{sk}}u_{sk}W'(1, \infty, 0) + \cdots]. \tag{5.7}$$

In the absence of skewness the amplitude $W_3(1, \infty, 0)$ of the leading term is equal to zero and the third moment scales as $W_3 \simeq AL^x$ with $x = 3\alpha + y_{sk}$. Figure 4 demonstrates that stationary-state skewness is indeed absent at the KPZ fixed point. The exponent $x = 0.4 \pm 0.1$ suggests that the skewness crossover exponent is equal to $y_{sk} = -1$ (the nearest integer). The amplitude A of the third moment, see figure 5, must be proportional to u_{sk}. The local measure for the skewness, ρ_{sk}, in figure 3 should be approximately proportional to u_{sk} as well. We calculated the ratio A/ρ_{sk} at several points in the phase diagram, and find it indeed to be almost a constant.

In conclusion, in this paper we study the phase diagram of the RSOS growth model in $(1 + 1)$ dimensions. Its phase diagram contains a 4D subspace in which the stationary state is completely uncorrelated. Familiar models, such as the KK model, lie outside this subspace. Their stationary states contain additional features, such as skewness. Stationary-state skewness diverges with system size, but such that the scaling properties are still described by the KPZ fixed point at zero skewness. The skewness crossover exponent is equal to $y_{sk} \simeq -1$.

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